# MATHEMATICAL FOUNDATIONS OF DATA ANALYSIS

## UNIT V

## SINGULAR VALUE DE COMPOSITION OF A MATRIX

**Definition: Singular Value Decomposition of a matrix:**

**A Singular Value Decomposition (SVD)** of an m × n matrix A of rank r is a factorization $\mathbf{A = U\Sigma V^T}$ where **U and V are orthogonal** and $\mathbf{\Sigma} = \begin{bmatrix} \boldsymbol{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ m×n in block form where $\mathbf{D = diag\ (d_1, d_2, ..., d_r)}$ where each $d_i > 0$, and $r \leq m$ and $r \leq n$.

**Note1 :** If $A = U\Sigma V^T$ is any SVD for A as then:

1. r = rank A.

2. The numbers $d_1, d_2, ..., d_r$ are the singular values of $A^TA$ in some order.

**Note 2 :**

Let A be a real m×n matrix. Then:

1. The eigen values of $A^TA$ and $AA^T$ are **real and non-negative.**

2. $A^TA$ and $AA^T$ have the same set of **positive eigen values**.

**Definition: Singular values of the matrix A**

Let A be a real m×n matrix. Let λ be an **eigenvalue of $\mathbf{A^TA}$**, with non zero eigenvectors $q_i \in R^n$. Then the **real numbers** $\boldsymbol{\sigma_i} = \sqrt{\boldsymbol{\lambda_i}} = ||\boldsymbol{Aq_i}||$ for i = 1, 2, ..., n, are called the **singular values of the matrix A** .

**Definition: Singular matrix of A**

Let A be a real, m×n matrix of rank r, with **positive singular values** $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $\sigma_i = 0$ if i > r. Define: $\mathbf{D_A = diag\ (\sigma_1, \sigma_2\ ..., \sigma_r)}$ and $\Sigma_A = \begin{bmatrix} D_A & 0 \\ 0 & 0 \end{bmatrix}_{m\times n}$ Here $\Sigma_A$ is in **block form** and is called the **Singular matrix of A.**

**Definition:** Two subspaces associated with a matrix A having m rows and n columns.

**im A = { Ax | $x \in R^n$ } and col A = span {a | a is a column of A}.**

Then **im A** is called the **image of A** (so named because of the linear transformation $R^n \to R^m$ with $x \to Ax$); and **col A** is called the **column space of A**.

Note : **im A = col A.**

**Definition: Singular Value Decomposition (SVD) of A**

**Definition:** Let A be a real m×n matrix, and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ be the positive **singular values of A**. Then **r is the rank of A** and we have the factorization **A = PΣ$_A$Q$^T$** where **P and Q are orthogonal matrices.** The factorization **A = PΣ$_A$Q$^T$ ,** where **P and Q are orthogonal matrices**, is called a **Singular Value Decomposition (SVD) of A.** This decomposition is not unique.

**Reference:**

**https://math.emory.edu/~lchen41/teaching/2020_Fall/Section_8-6.pdf**

**Procedure for finding Singular Value Decomposition (SVD) of the given matrix $A$**

**Solution:**

Step 1: To find $A^T A$

Step 2: To find Eigen values (Characteristic values) of $A^T A$.

Use $|A^T A - \lambda I| = 0$

Step 3: To find Eigen vectors (Characteristic vectors) Eigen vectors of $A^T A$ , corresponding unit Eigen vectors $q_1$, $q_2$ and $q_3$.

Use the Characteristic equations formula $(A^T A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Step 4: To find orthogonal matrix $Q = (q_1 \ \ q_2 \ \ q_3)$

Step 5: To find Singular values of $A^T A$.

Use the formula $\sigma_i = \sqrt{\lambda_i}$ i = 1,2 3. $\sigma_i$ is called the Singular value of the matrix $A^T A$

Step 6: To find rank of the given matrix A

Step 7: To find singular matrix of A denoted by $\sum_A$

$$\sum_A = \begin{pmatrix} \mathrm{D_A} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{Where } \mathrm{D_A} = \mathrm{diag}\,(\sigma_1 \ \ \sigma_2 \ \ \sigma_3 \ldots)$$

Step 8: To find $2 \times 2$ Orthogonal matrix $P = (p_1 \quad p_2)$ where $p_1$ and $p_2$ are normalisers of $Aq_1$ and $Aq_2$ respectively.

Step 9: Singular Value Decomposition (SVD) f or A is obtained by using the formula

$$A = P \sum_A Q^T$$

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**Problem:**

**Find Singular Value Decomposition (SVD) for the matrix** $\boldsymbol{A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}}$

**Solution:**

Let the given matrix be $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

**Step 1: To find** $\boldsymbol{A^T A}$

$$A^T A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

**Step 2: To find Eigen values (Characteristic values ) of** $\boldsymbol{A^T A}$

$$|A^T A - \lambda I| = \mathbf{0}$$

$$\Rightarrow \left| \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \mathbf{0}$$

$$\Rightarrow \left| \begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} \right| = \mathbf{0}$$

$$\Rightarrow (2-\lambda)(1-\lambda)(1-\lambda) - (1-\lambda) - (1-\lambda) = \mathbf{0}$$

$$\Rightarrow (1-\lambda)\,[\,(2-\lambda)(1-\lambda) - \mathbf{1} - \mathbf{1}\,] = \mathbf{0}$$

$$\Rightarrow (1-\lambda)\,[\lambda^{\mathbf{2}} - \mathbf{3}\,\lambda + 2 - 2\,] = \mathbf{0}$$

$$\Rightarrow (1-\lambda)\,[\lambda^{\mathbf{2}} - \mathbf{3}\,\lambda] = \mathbf{0}$$

$$\Rightarrow (1-\lambda)\,(\lambda - 3)\lambda = \mathbf{0}$$

$$\Rightarrow \lambda = 3\,, \lambda = 1 \; and \; \lambda = 0$$

**Let** $\boldsymbol{\lambda_1 = 3}$ $\boldsymbol{\lambda_2 = 1}$ **and** $\boldsymbol{\lambda_3 = 0}$

**Step 3: To find Eigen vectors (Characteristic vectors) Eigen vectors of** $\boldsymbol{A^T A}$ **, corresponding unit Eigen vectors** $\boldsymbol{q_1}$**,** $\boldsymbol{q_2}$ **and** $\boldsymbol{q_3}$**.**

Characteristic equations is $(A^T A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow (A^T A - \lambda I)X = 0$$

$$\Rightarrow \begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(2-\lambda)x_1 - x_2 + x_3 = 0$$

$$-x_1 + (1-\lambda)\,x_2 + 0\,x_3 = 0$$

$$x_1 + 0\,x_2 + (1-\lambda)\,x_3 = 0$$

Put $\boldsymbol{\lambda} = \mathbf{3}$ in the above first two equations we get

$$-x_1 - x_2 + x_3 = 0$$

$$-x_1 - 2\,x_2 + 0\,x_3 = 0$$

| $x_2$ | $x_3$ | $x_1$ | $x_2$ |
|---|---|---|---|
| -1 | 1 | -1 | -1 |
| -2 | 0 | -1 | 0 |

$$\frac{x_1}{0+2} = \frac{x_2}{-1-0} = \frac{x_3}{0-1}$$

$$\Longrightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

Therefore Eigen vector corresponding to the Eigen value $\lambda = 3$ is $\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

Let $q_1$ be the unit Eigen vector corresponding to the Eigen value $\lambda = 3$.

$$q_1 = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \textit{ the length of this vector is } \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{6}$$

Next Put $\boldsymbol{\lambda} = \mathbf{1}$ in the above first two equations we get

$$1\,x_1 - x_2 + x_3 = 0$$

$$-x_1 + 0\,x_2 + 0\,x_3 = 0$$

| $x_2$ | $x_3$ | $x_1$ | $x_2$ |
|---|---|---|---|
| -1 | 1 | 1 | -1 |
| 0 | 0 | -1 | 0 |

$$\frac{x_1}{0+0} = \frac{x_2}{-1-0} = \frac{x_3}{0-1}$$

$$\Longrightarrow \frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1} \quad \Longrightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore Eigen vector corresponding to the Eigen value $\lambda = 1$ is $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Let $q_2$ be the unit Eigen vector corresponding to the Eigen value $\lambda = 1$.

$$\text{Therefore } \boldsymbol{q_2} = \begin{pmatrix} \mathbf{0} \\ \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \\ \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \end{pmatrix} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix} \textit{ the length of this vector is } \sqrt{0^2 + (1)^2 + (1)^2} = \sqrt{2}$$

Next Put $\boldsymbol{\lambda} = \mathbf{0}$ in the above first two equations we get

$$2\,x_1 - x_2 + x_3 = 0$$

$$-x_1 + 1\,x_2 + 0\,x_3 = 0$$

| $x_2$ | $x_3$ | $x_1$ | $x_2$ |
|---|---|---|---|
| -1 | 1 | 2 | -1 |
| 1 | 0 | -1 | 1 |

$$\frac{x_1}{0-1} = \frac{x_2}{-1-0} = \frac{x_3}{2-1}$$

$$\Longrightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Therefore Eigen vector corresponding to the Eigen value $\lambda = 0$ is $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

Let $q_3$ be the unit Eigen vector corresponding to the Eigen value $\lambda = 0$.

Therefore $\boldsymbol{q_3} = \begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ $\quad$ *the length of this vector is* $\sqrt{(-1)^2 + (-1)^2 + (1)^2} = \sqrt{3}$

**Step 4: To find orthogonal matrix $\boldsymbol{Q} = (\boldsymbol{q_1}\ \boldsymbol{q_2}\ \boldsymbol{q_3})$**

The **orthogonal matrix** $Q$ is

$$Q = [\boldsymbol{q_1}\ \boldsymbol{q_2}\ \boldsymbol{q_3}] = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

By taking $\frac{1}{\sqrt{6}}$ outside we get $Q = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & 0 & -\sqrt{2} \\ -1 & \sqrt{3} & -\sqrt{2} \\ 1 & \sqrt{3} & \sqrt{2} \end{pmatrix}$

**Step 5: To find Singular values of $\boldsymbol{A^T A}$.**

Use the formula $\sigma_i = \sqrt{\lambda_i}$ i = 1,2 3. $\sigma_i$ is called the Singular value of the matrix $A^T A$.

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$ $\qquad$ $\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$ $\qquad$ $\sigma_3 = \sqrt{\lambda_3} = \sqrt{0} = 0$

**Step 6: To find rank of the given matrix A**

$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$. Here order of A is $2 \times 3$. Therefore Rank of A, $\rho\,(A) \leq 2$

Now , $\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 \neq 0$. Therefore $\rho\,(A) = 2$.

**Step 7: To find singular matrix of A denoted by $\boldsymbol{\sum_A}$**

$$\textstyle\sum_A = \begin{pmatrix} \mathrm{D_A} & 0 \\ 0 & 0 \end{pmatrix} \text{ Where } \mathbf{D_A} = \mathbf{diag}\,(\boldsymbol{\sigma_1}\ \boldsymbol{\sigma_2}\ \boldsymbol{\sigma_3}) = \mathrm{diag}\,(\sqrt{3}\ \ 1\ \ 0)$$

Since the Rank of A is 2 , we consider only two singular values $\sigma_1$ and $\sigma_2$.Therefore Singular matrix of A is

$$\Sigma_A = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

**Step 8: To find $2 \times 2$ Orthogonal matrix $P = (p_1 \quad p_2)$ where $p_1$ and $p_2$ are normalisers of $Aq_1$ and $Aq_2$ respectively.**

$$\text{Now } \boldsymbol{Aq_1} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{6}} \\ -\frac{3}{\sqrt{6}} \end{pmatrix} = \frac{3}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\boldsymbol{p_1} = \text{ Normaliser of } Aq_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\text{Now } \boldsymbol{Aq_2} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{p_2} = \text{ Normaliser of } Aq_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Therefore **Orthogonal matrix**

$$\boldsymbol{P} = (\boldsymbol{p_1} \quad \boldsymbol{p_2}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

**Step 9:** Singular Value Decomposition (SVD) f or A is obtained by using the formula

$$A = P \Sigma_A Q^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & 0 & -\sqrt{2} \\ -1 & \sqrt{3} & -\sqrt{2} \\ 1 & \sqrt{3} & \sqrt{2} \end{pmatrix}^T$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & -1 & 1 \\ 0 & \sqrt{3} & \sqrt{3} \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2} \end{pmatrix}$$